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Instanton calculations in the β-deformed AdS/CFT correspondence

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Abstract: We consider non-perturbative effects in the β-deformed $\mathcal{N} = 4$ supersymmetric gauge theory in the context of the AdS/CFT correspondence. We concentrate on certain types of the n-point correlation functions of the Yang-Mills operators which correspond to the lowest Kaluza-Klein modes propagating on the dual supergravity background found by Lunin and Maldacena in [1]. In particular, we calculate all multi-instanton contributions to these correlators in the β-deformed SYM and find a compelling agreement with the results expected in supergravity.

Keywords: AdS-CFT Correspondence, 1/N Expansion, Solitons Monopoles and Instantons.

Contents

1. Introduction

β-deformations of the $\mathcal{N} = 4$ supersymmetric Yang-Mills define a family of conformally-invariant four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. Remarkably, these β-deformed theories mirror many non-trivial characteristic features of the $\mathcal{N} = 4$ SYM, including the S-duality, the AdS/CFT correspondence and the perturbative large-N equivalence to the parent $\mathcal{N} = 4$ theory, thus providing a continuous class of interesting generalizations of the $\mathcal{N} = 4$ SYM. One may expect that by studying properties of the β-deformed theories and, in particular, the dependence of observables on the continuous deformation parameter β we can understand better the gauge dynamics of this class of theories and of the $\mathcal{N} = 4$ as well.

Dualities between gauge and string theories have been studied intensively for more than three decades. The AdS/CFT correspondence formulated in [2−4] provides a concrete realization of such a duality. In its original formulation, the AdS/CFT duality relates the string theory on a curved background $AdS_5 \times S^5$ to the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory living on the boundary of AdS_5. Understanding this duality in detail beyond the BPS and the near BPS limits remains a challenge mainly due to the fact that one has to deal with the weak-to-strong coupling correspondence.

The AdS/CFT duality extends to the β-deformed theories [1] where it relates the β-deformed $\mathcal{N} = 4$ SYM and the supergravity on the deformed $AdS_5 \times \tilde{S}^5$ background. The gravity dual was found by Lunin and Maldacena in ref. [1], and this provides a precise formulation of the AdS/CFT duality in the deformed case, which can be probed and studied using instanton methods developed earlier for the $\mathcal{N} = 4$ case in [5 − 8] and in [9 − 12].

Several perturbative calculations in the β-deformed theories were carried out recently in [13 − 16] where it was noted that there are many similarities between the deformed and the undeformed theories which emerge in the large number of colours limit. In particular, in [16] it was shown that in perturbation theory there is a close relation between the scattering amplitudes in the β_R-deformed and in the original $\mathcal{N} = 4$ theory. This correspondence holds in the large-N limit and to all orders in (planar) perturbation theory. It states that for real values of β all amplitudes in the β-deformed theory are given by the corresponding $\mathcal{N} = 4$ amplitudes times an overall β-dependent phase factor. The phase factor depends only on the external legs and is easily determined for each class of amplitudes [16]. It follows from these considerations [16] that the recent proposal of Bern, Dixon and Smirnov [17] which determines all multi-loop MHV planar amplitudes in the $\mathcal{N} = 4$ superconformal Yang-Mills theory can be carried over to a wider family of gauge theories obtained by real β-deformations of the $\mathcal{N} = 4$ Yang-Mills.

The purpose of this paper is to consider non-perturbative instanton effects in the β-deformed theories and in the context of the AdS/CFT correspondence. The Lunin-Maldacena example [1] of the AdS/CFT duality relates the large-N limit of the β-deformed $\mathcal{N} = 4$ gauge theory to the type IIB string theory on the appropriately β-deformed $AdS_5 \times \tilde{S}^5$ background. The β-deformed gauge theory is living on the 4-dimensional boundary of the AdS_5 space. It is expected that each chiral primary operator (and its superconformal descendants) in this boundary conformal theory corresponds in supergravity to a particular Kaluza-Klein mode on the deformed sphere $\tilde{S}^5$. In this paper we will consider only the operators which correspond to the supergravity states which do not depend on $\tilde{S}^5$ coordinates, i.e. which are the lowest Kaluza-Klein modes on the deformed sphere. Furthermore, to simply the derivations, we will restrict ourselves to a particular class class of such operators, considered previously in [5, 8], and to the minimal in n classes of their correlators G_n.

Following the approach developed in ref. [8], we will evaluate all multi-instanton contributions to these correlation functions in the appropriate large-N scaling limit[1] and to the leading non-vanishing order in perturbation theory around instantons. We will show that these correlation functions in the β-deformed $\mathcal{N} = 4$ SYM are in precise correspondence with the supergravity expectations. More precisely, we will be able to reproduce a class of leading higher-derivative corrections to the supergravity effective action, S_{eff}, from Yang-Mills instantons. In particular, we will see that the multi-instanton contributions to G_n will reconstruct the appropriate moduli forms $f_n(\tau, \bar{\tau})$ present in the S_{eff}. This is a particularly non-trivial observation since the dilaton-axion τ parameter in the *deformed*

[1]The large-N limit appropriate for the comparison with the supergravity solution of [1], will also require that the deformation parameter β is kept small. Hence, starting from section 5 we will take $N \to \infty$ and $\beta \to 0$.

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supergravity solution is not anymore equal to the complexified coupling τ_0 of the gauge theory. The dilaton ϕ is, in fact, a non-trivial function of the coordinates μ_i on the deformed sphere [1]

$$e^{\phi} = G^{1/2}(\beta;\mu_1,\mu_2,\mu_3)\cdot\frac{g^2}{4\pi} \tag{1.1}$$

Here g^2 is the Yang-Mills coupling constant and G is the function appearing in the Lunin-Maldacena solution in eqs. (2.2)-(2.4) below. It will turn out, that in the β-deformed $\mathcal{N}=4$ SYM, a proper inclusion of instanton collective coordinates will effectively upgrade the usual exponent of the k-instanton action $e^{2\pi i k\tau_0}$ into the required expression $e^{2\pi i k\tau}$.

The rest of our findings parallels those in ref. [8]. We shall find that in the appropriately taken large-N scaling limit, the k-instanton collective coordinate measure has a geometry of a single copy of the 10-dimensional space $AdS_5\times\tilde{S}^5$. We shall also observe that this k-instanton measure includes the partition function of the $SU(k)$ matrix model, thus matching the description of the D-instantons as D(-1) branes in string theory. In the appendix we show that the full Yang-Mills k-instanton integration measure in the deformed theory is equivalent to the partition function of k D-instantons in the corresponding string theory where β-deformations are introduced via star products.

In most of what follows we will consider the transformations with a real deformation parameter $\beta=\beta_R$. The generalization to the case of complex deformations will be carried out only in the end in section 8. As one would have anticipated, our results and the matching between the supergravity and the gauge theory expressions found at real values of β also persist for complex deformations. One reason for this is the fact [1] that the backgrounds with complex β can be generated by performing $SL(2,R)_s$ transformations on the solutions with real β. At the same time, in gauge theory, β should transform [18] as a modular form under the $SL(2,Z)_s$ S-duality. Some other related aspects of the β-deformed gauge theory have been studied previously in refs. [19 – 32]

The β-deformations of the $\mathcal{N}=4$ supersymmetric gauge theory are described by the superpotential

$$ig\,\mathrm{Tr}(e^{i\pi\beta_R}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta_R}\Phi_1\Phi_3\Phi_2)\,, \tag{1.2}$$

where Φ_i are chiral $\mathcal{N}=1$ superfields. The resulting superpotential preserves the $\mathcal{N}=1$ supersymmetry of the original $\mathcal{N}=4$ SYM and leads to a theory with a global $U(1)\times U(1)$ symmetry (in addition to the usual $U(1)_R$ R-symmetry of the $\mathcal{N}=1$ susy) [1]

$$\begin{aligned} U(1)_1: &\qquad (\Phi_1,\Phi_2,\Phi_3)\to(\Phi_1,e^{i\varphi_1}\Phi_2,e^{-i\varphi_1}\Phi_3)\\ U(1)_2: &\qquad (\Phi_1,\Phi_2,\Phi_3)\to(e^{-i\varphi_2}\Phi_1,e^{i\varphi_2}\Phi_2,\Phi_3) \end{aligned} \tag{1.3}$$

It is known [15] that (1.2) describes an exactly marginal deformation of the theory in the limit of large number of colors. (In a more general case of complex deformations, for the theory to remain conformal one needs to satisfy the so-called Leigh-Strassler constraint [33]. For the real β_R case this constraint is trivial in the large N limit.)

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2. Supergravity dual

The gravity dual of the β-deformed $\mathcal{N} = 4$ gauge theory was identified by Lunin and Maldacena in [1]. The $U(1) \times U(1)$ global symmetry (1.3) of the β-deformed SYM plays an important rôle in this approach. One starts with the original $AdS_5 \times S^5$ background and compactifies it on a two-torus in such a way that the isometries of the torus match with the global $U(1) \times U(1)$ symmetry in gauge theory. The idea [1] is then to use the $SL(2,R)$ symmetry of type IIB supergravity compactified along the two-torus to generate a new solution of the supergravity equations. The $SL(2,R)$ transformation acts on the complex parameter $\tau = B_{12} + i\sqrt{g}$ of the original gravitational theory. Here B_{12} is the NS-NS two-form field along the torus directions, and g is the determinant of the metric on the torus. The $SL(2,R)$ acts on this torus τ-parameter as follows

$$\tau \longrightarrow \frac{\tau}{1 + \beta_R \tau} \tag{2.1}$$

The geometry obtained in this way is (in the string frame) the product of $AdS_5 \times \tilde{S}^5$, where $\tilde{S}^5$ is a deformed five-sphere. In what follows we write down only the part of the supergravity solution which will be relevant for our purposes,[2]

$$ds^2_{str} = R^2 \left[ds^2_{AdS_5} + \sum_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 G \mu_1^2 \mu_2^2 \mu_3^2 (\sum_i d\phi_i)^2 \right] , \tag{2.2}$$

$$e^{\phi} = e^{\phi_0} G^{1/2} , \tag{2.3}$$

$$G^{-1} = 1 + \hat{\gamma}^2 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2) , \qquad \hat{\gamma} := R^2 \beta_R , \qquad R^4 = 4\pi e^{\phi_0} N \tag{2.4}$$

The deformed five-sphere in the supergravity solution above is parameterized by the three radial variables μ_i, which satisfy the condition $\sum_{i=1}^3 \mu_i^2 = 1$, and the three angles ϕ_i.

The complete supergravity solution (which is valid for generic complex values of β) can be found in the original paper [1]. In addition to the five-form field F_5 already present in the $AdS_5 \times S^5$ geometry, the solution also includes the NS-NS two-form potential B_2 and the RR potential C_2.

It is important to note that the dilaton ϕ is no longer constant, but depends on the coordinates μ_i of the deformed five-sphere. The constant parameter is ϕ_0 which has the meaning of the dilaton of the parent undeformed solution, and it maps to the coupling constant of the dual gauge theory. However, it is ϕ and not ϕ_0 which plays the røle of the dilaton in the deformed supergravity solution. The dilaton ϕ and the axion C are assembled in the standard way into a complex τ

$$\tau = ie^{-\phi} + C \tag{2.5}$$

Equation (2.3) relates this τ to the complexified coupling constant of the dual gauge theory,

$$\tau_0 = ie^{-\phi_0} + C = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} . \tag{2.6}$$

[2] As mentioned earlier we also restrict to real values of $\beta = \beta_R$ which was called γ in [1]. We will comment on the complex deformations in section 8.

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In summary, the dictionary between the parameters of the deformed Yang-Mills theory and type IIB superstring theory on $AdS_5 \times S^5_\gamma$ is as follows:

$$e^{-\phi}\, G^{1/2} \;=\; e^{-\phi_0} = \frac{4\pi}{g^2}\,, \qquad C \;=\; \frac{\theta}{2\pi} \tag{2.7}$$

$$R^2 \;=\; \sqrt{g^2\, N} \tag{2.8}$$

and R is the radius of the AdS_5 space in units of $\sqrt{\alpha'}$. The supergravity background is a valid approximation to string theory in the small curvature regime [1]:

$$R \gg 1\,, \qquad R\beta_R \ll 1\,. \tag{2.9}$$

In terms of the gauge theory variables, the appropriate limit to consider is

$$g^2 N \gg 1\,, \qquad N \gg 1\,, \qquad \beta_R \ll 1\,. \tag{2.10}$$

In the above, the $N \gg 1$ condition arises from the fact that the $SL(2,Z)$ duality can be used to map large string couplings to values which are not large.

As is well known, the supergravity action of type IIB theory is invariant under the non-compact symmetry group $SU(1,1) \sim SL(2,R)$. The action of this symmetry leaves the metric invariant, but acts upon the dilaton-axion field τ of eq. (2.5)

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \qquad ad - bc = 1,\;\; a,b,c,d \in R. \tag{2.11}$$

The string theory is invariant only under the $SL(2,Z)$ subgroup of the $SL(2,R)$. This implies that the string theory effective action $S_{\rm IIB}$ must be invariant under the $SL(2,Z)$ S-duality transformation.

The string effective action $S_{\rm IIB}$ is related via the AdS/CFT holographic formula [3, 4] to correlation functions in the gauge theory,

$$\exp - S_{\rm IIB}\,[\Phi_{\mathcal{O}}; J] \;=\; \left\langle \exp \int d^4x\, J(x)\mathcal{O}(x) \right\rangle. \tag{2.12}$$

Here $\Phi_{\mathcal{O}}$ are Kaluza-Klein modes of the supergravity fields which are dual to composite gauge theory operators $\mathcal{O}$. The boundary conditions of the supergravity fields are set by the gauge theory sources on the boundary of AdS_5 via $\Phi_{\mathcal{O}}(x) \,\propto\, J(x)$.

Constructing all the Kaluza-Klein modes on the deformed five-sphere is a non-trivial task, hence, in this paper we are restricting ourselves to the lowest Kaluza-Kein modes which do not depend on the coordinates of the deformed sphere.

D-instanton contributions in supergravity arise as $(\alpha')^3$ corrections [9] to the classical IIB theory. The D-instanton contribution to an n-point correlator G_n comes from a tree level Feynman diagram with one vertex located at a point $(x_0, \rho, \hat{\Omega})$ in the bulk of $AdS_5 \times \tilde{S}^5$. The diagram also has n external legs connecting the vertex to operator insertions on the boundary. There is a bulk-to-boundary propagator associated with each external leg [4, 34]. For example, an $SO(6)$ singlet scalar free field of mass m on AdS_5 has the bulk-to-boundary propagator

$$K_\Delta(x_0, \rho; x, 0) = \frac{\rho^\Delta}{(\rho^2 + (x - x_0)^2)^\Delta}\,, \tag{2.13}$$

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where $(mL)^2 = \Delta(\Delta - 4)$. At leading order beyond the Einstein-Hilbert term in the derivative expansion, the IIB effective action is expected to contain [35, 9] an $\mathcal{R}^4$ term[3]

$$(\alpha')^{-1} \int d^{10}x \sqrt{-g_{10}}\, e^{-\phi/2} f_4(\tau, \bar{\tau})\, \mathcal{R}^4 \tag{2.14}$$

as well as its superpartners, including a totally antisymmetric 16-dilatino effective vertex of the form [36, 37]

$$(\alpha')^{-1} \int d^{10}x \sqrt{-g_{10}}\, e^{-\phi/2} f_{16}(\tau, \bar{\tau})\, \Lambda^{16} \;+\; \text{H.c.} \tag{2.15}$$

The dilatino Λ is a complex chiral $SO(9,1)$ spinor which transforms under the local $U(1)$ symmetry with the charge $q_\Lambda = 3/2$. Under the $SL(2,Z)$ transformations (2.11) all fields Φ are multiplied by a (discrete) phase,

$$\Phi \longrightarrow \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{-q_\Phi/2} \Phi\,, \tag{2.16}$$

and the charge q_Φ for the dilatino is 3/2 and for the $\mathcal{R}$ field it is zero.

Equations (2.14)–(2.15) are written in the string frame with the coefficients $f_n(\tau, \bar{\tau})$ being the modular forms of weights $((n-4), -(n-4))$ under the $SL(2,Z)$ transformations (2.11),

$$f_n(\tau, \bar{\tau}) \;:=\; f^{\,(n-4),-(n-4)}(\tau, \bar{\tau}) \;\longrightarrow\; \left(\frac{c\tau + d}{c\bar{\tau} + d}\right)^{n-4} f^{\,(n-4),-(n-4)}(\tau, \bar{\tau})\,. \tag{2.17}$$

The modular properties of f_n precisely cancel the phases of fields in (2.16) acquired under the $SL(2,Z)$. Thus the full string effective action is invariant under the $SL(2,Z)$ and this modular symmetry ensures the S-duality of the type IIB superstring.

The modular forms f_n have been constructed by Green and Gutperle in [35]. In the weak coupling expansion the expressions for f_n contain an infinite sum of exponential terms

$$e^{-\phi/2} f_n \ni \sum_{k=1}^{\infty} \text{const} \cdot \left(\frac{k}{G^{1/2}\, g^2}\right)^{n-7/2} e^{2\pi i k \tau} \sum_{d|k} \frac{1}{d^2}\,, \tag{2.18}$$

In the original undeformed $\mathcal{N} = 4$ scenario, $\tau = \tau_0$ and the modular forms f_n in the string effective action can be thought of as functions of the gauge coupling constants τ_0 and $\bar{\tau}_0$. In this case, each of the terms in the expression above must correspond to a contribution of an instanton of charge k. On the other hand, the k-instanton contributions can be independently calculated directly in gauge theory. These calculations have been performed in [5, 6] at the 1-instanton level and in [7, 8] for the general k-instanton case. Remarkably, the SYM results of [5−8] tuned out to be in precise agreement with the supergravity predictions for the effective action eqs. (2.14)–(2.18) and in eq. (2.20) below.

The goal of the present paper is to attempt to reproduce these results in the β-deformed case. Hence we want to interprete the sum on the right hand side of (2.18) again as coming

[3] Here, $\mathcal{R}^4$ denotes a particular contraction [35] of four ten-dimensional Riemann tensors.

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from multi-instantons in gauge theory. We see that, at least potentially, there is a puzzle in this interpretation as the Yang-Mills k-instantons are expected to contribute factors proportional to $e^{2\pi i k\tau_0}$ rather than to $e^{2\pi i k\tau}$. In the rest of the paper when we perform an explicit k-instanton calculation in the β-deformed SYM we will find the resolution of this puzzle.

The main object of interest for us are the n-point correlation functions of certain composite operators in the β-deformed SYM. We can consider the same classes of the operators as in [5, 8, 12] which correspond to the lowest KK-modes in supergravity. Specifically we will analyze the gauge-invariant chiral correlators G_n, $n = 16$, 8 or 4, defined by:

$$G_{16} = \langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_{16})\rangle\,, \quad \mathcal{O} := \Lambda^A_\alpha = g^{-2}\sigma^{mn}{}_\alpha{}^\beta \,\mathrm{tr}_N\, F_{mn}\,\lambda^A_\beta\,, \tag{2.19a}$$
$$G_8 = \langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_8)\rangle\,, \quad \mathcal{O} := \mathcal{B}^{[AB]}_{mn} = g^{-2}\,\mathrm{tr}_N\left(\lambda^A\sigma_{mn}\lambda^B + 2iF_{mn}\Phi^{AB}\right), \tag{2.19b}$$
$$G_8 = \langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_8)\rangle\,, \quad \mathcal{O} := \mathcal{E}^{(AB)} = g^{-2}\,\mathrm{tr}_N\left(\lambda^A\lambda^B + t^{(AB)+}_{[abc]}\phi^a\phi^b\phi^c\right), \tag{2.19c}$$
$$G_4 = \langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_4)\rangle\,, \quad \mathcal{O} := \mathcal{Q}^{ab} = g^{-2}\,\mathrm{tr}_N\left(\phi^a\phi^b - \tfrac{1}{6}\delta^{ab}\phi^c\phi^c\right), \tag{2.19d}$$

where t in eq. (2.19c) is a numerical tensor. In the notation of ref. [12] these correlators were called the minimal ones. The non-minimal correlators G_n with higher n were considered in [12] in the context of the original $\mathcal{N} = 4$ AdS/CFT correspondence. In the present paper we will concentrate on the minimal case above, and paying particular attention to the correlators in (2.19a) and (2.19b).

The AdS/CFT holographic relation then predicts that these correlators must lead on the supergravity side to the following expressions:

$$G_n \sim (\alpha')^{-1}\int d^4x\,\frac{d\rho}{\rho^5}\int G\,d^5\hat{\Omega}\;t_n\,e^{-\phi/2}\,f_n(\tau,\bar{\tau})\prod_{i=1}^{n}K(x_0,\rho;x_i,0) \tag{2.20}$$

Here $Gd^5\hat{\Omega}$ represents the volume form on the $\tilde{S}^5$ and each K denotes the bulk-to-boundary propagator which corresponds to each particular operator in (2.19a)–(2.19d). Various index contractions between the n states (propagators) are schematically represented by a tensor t_n. We want to verify the above relations using multi-instanton calculations in the β-deformed SYM theory. As in ref. [8] it will actually be sufficient for this purpose to concentrate on the multi-instanton partition function. The correlators can be obtained from the latter by inserting in it the operators calculated on the instanton solution.

3. Marginal β-deformations of $\mathcal{N} = 4$ SYM

The β-deformed Yang-Mills is an $\mathcal{N} = 1$ supersymmetric conformal gauge theory with a global $U(1)\times U(1)$ symmetry (1.3). Lunin and Maldacena have pointed out [1] that the β_R-deformation in (1.2) can be understood as arising from introducing the star products between the fields in the $\mathcal{N} = 4$ Lagrangian,

$$f * g \equiv e^{i\pi\beta_R(Q_1^f Q_2^g - Q_2^f Q_1^g)} fg \tag{3.1}$$

Here $(Q_1^{\rm field}, Q_2^{\rm field})$ are the $U(1)_1 \times U(1)_2$ charges of the fields (f or g). The values of the charges for component fields are read from (1.3):

$$\Phi_1\,,\ \lambda_1\ :\qquad (Q_1\,,Q_2) = (0\,,-1) \tag{3.2}$$
$$\Phi_2\,,\ \lambda_2\ :\qquad (Q_1\,,Q_2) = (1\,,1) \tag{3.3}$$
$$\Phi_3\,,\ \lambda_3\ :\qquad (Q_1\,,Q_2) = (-1\,,0) \tag{3.4}$$
$$A_m\,,\ \lambda_4\ :\qquad (Q_1\,,Q_2) = (0\,,0) \tag{3.5}$$

and for the conjugate fields ($\bar\Phi_i$ and $\bar\lambda_i$) the charges are opposite.

The component Lagrangian of the β_R-deformed theory is easily read from the $\mathcal{N}=4$ Lagrangian

$$\mathcal{L} = \ {\rm Tr}\Bigg(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\bar\Phi^i)(D_\mu\Phi_i) - \frac{g^2}{2}[\Phi_i,\Phi_j]_*[\bar\Phi^i,\bar\Phi^j]_* + \frac{g^2}{4}[\Phi_i,\bar\Phi^i][\Phi_j,\bar\Phi^j] \tag{3.6}$$
$$+\lambda_A\sigma^\mu D_\mu\bar\lambda^A - ig([\lambda_4,\lambda_i]\bar\Phi^i + [\bar\lambda^4,\bar\lambda^i]\Phi_i) + \frac{ig}{2}(\epsilon^{ijk}[\lambda_i,\lambda_j]_*\Phi_k + \epsilon_{ijk}[\bar\lambda^i,\bar\lambda^j]_*\bar\Phi^k)\Bigg).$$

In the above equation the star products (3.1) are used for fields charged under the $U(1)_1 \times U(1)_2$. We have also used the fact that the star product is trivial between two fields which have opposite $U(1)_1 \times U(1)_2$ charges. We have also introduced the β_R-deformed commutator of fields which is simply

$$[f_i, g_j]_* := \ f_i * g_j - g_j * f_i = \ e^{i\pi\beta_{ij}}\, f_i g_j - \ e^{-i\pi\beta_{ij}}\, g_j f_i\,, \tag{3.7}$$

and β_{ij} is defined as

$$\beta_{ij} = -\beta_{ji}\,, \quad \beta_{12} = \ -\beta_{13} = \ \beta_{23} := \ \beta_R\,. \tag{3.8}$$

More generally, and for future reference we also define a 4×4 deformation matrix β_{AB} with $A, B = 1,\ldots,4$

$$\beta_{AB} = -\beta_{BA}\,, \quad \beta_{4i} = 0\,, \quad \beta_{12} = \ -\beta_{13} = \ \beta_{23} := \ \beta_R\,. \tag{3.9}$$

The component Lagrangian in the form (3.6) is well-suited for tracing the β_R-dependence in perturbative calculations and it was utilized in [16].

For carrying out multi-instanton calculations in the formalism of [8, 11] it is more convenient to switch to a different basis for scalar fields. We assemble the three complex scalars Φ_i into an adjoint-valued antisymmetric tensor field $\Phi^{AB}(x)$, subject to a specific reality condition:

$$\tfrac{1}{2}\epsilon_{ABCD}\,\Phi^{CD} \ = \ \bar\Phi^{AB}\,, \tag{3.10}$$

which implies that it transforms in the vector **6** representation of $SO(6)_R$ symmetry of the $\mathcal{N}=4$ SYM. In terms of the six real scalars ϕ^a it can be written as [8]

$$\Phi^{AB} = \frac{1}{\sqrt{8}}\bar\Sigma^{AB}_a\phi^a\,, \qquad \bar\Phi^{AB} = \ -\frac{1}{\sqrt{8}}\Sigma^{AB}_a\phi^a\,, \qquad a = 1,\ldots,6 \tag{3.11}$$

where the coefficients Σ_a^{AB} and $\bar{\Sigma}_a^{AB}$ are expressed in terms of the 't Hooft η-symbols:

$$\Sigma_{AB}^a = \big(\eta_{AB}^1, i\bar{\eta}_{AB}^1, \eta_{AB}^2, i\bar{\eta}_{AB}^2, \eta_{AB}^3, i\bar{\eta}_{AB}^3\big), \tag{3.12}$$
$$\bar{\Sigma}_a^{AB} = \big(-\eta_{AB}^1, i\bar{\eta}_{AB}^1, -\eta_{AB}^2, i\bar{\eta}_{AB}^2, -\eta_{AB}^3, i\bar{\eta}_{AB}^3\big)\,. \tag{3.13}$$

Here η and $\bar{\eta}$ are the selfdual and anti-selfdual 't Hooft symbols [38]:

$$\begin{aligned}
&\bar{\eta}_{AB}^c = \eta_{AB}^c = \epsilon_{cAB} \qquad A, B \in \{1,2,3\},\\
&\bar{\eta}_{4A}^c = \eta_{A4}^c = \delta_{cA},\\
&\eta_{AB}^c = -\eta_{BA}^c, \qquad \bar{\eta}_{AB}^c = -\bar{\eta}_{BA}^c.
\end{aligned} \tag{3.14}$$

The relation between the two bases of scalar fields is then given by

$$\begin{aligned}
\Phi_1 &= \frac{1}{\sqrt{2}}(\phi^1 + i\phi^2) = 2\,\bar{\Phi}^{23} = 2\,\Phi^{14}\\
\Phi_2 &= \frac{1}{\sqrt{2}}(\phi^3 + i\phi^4) = 2\,\bar{\Phi}^{31} = 2\,\Phi^{24}\\
\Phi_3 &= \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6) = 2\,\bar{\Phi}^{12} = 2\,\Phi^{34}
\end{aligned} \tag{3.15}$$

and their $U(1)_1 \times U(1)_2$ charges can be read off (3.2)–(3.4)

4. Instantons in the β-deformed $\mathcal{N} = 4$ SYM

In pure gauge theory, instantons obey the self-duality equation

$$F_{mn} = {}^*F_{mn} \tag{4.1}$$

The ADHM k-instanton [39 − 41] is the gauge configuration, A_m, which is the general solution of (4.1) with the topological charge k. When gauge fields are coupled to fermions and scalars, as in the $\mathcal{N} = 4$ SYM, one needs to consider the coupled classical Euler-Lagrange equations instead of (4.1). Instanton configurations then also include fermion and scalar-field components. Our goal, however, is not just to find classical solutions, but rather to calculate their quantum contributions to correlators G_n, which includes the effects of a perturbative expansion in the instanton background. The way to take the leading perturbations into account automatically is to modify the background configuration itself as explained in [8]; however, the instanton supermultiplet is then no longer an exact solution to the coupled equations of motion. In particular, k-instanton fermion components in the $\mathcal{N} = 4$ $SU(N)$ SYM are defined [8] to contain all of the $8kN$ fermion zero modes of the Dirac operator, $\bar{\not{\mathcal{D}}}^{\dot{\alpha}\alpha}$ and not just the 16 exact unlifted zero modes. Similarly, in the β-deformed theory, the same total of $8kN$ fermion zero modes will be included into the k-instanton supermultiplet, even though only 4 of them are exact in the theory with $\mathcal{N} = 1$ supersymmetry.

As a result, our strategy (see [8, section 2] and in [11, section 4] for more detail) is to solve Euler-Lagrange equations iteratively, order by order in the Yang-Mills coupling. In this paper we restrict our attention to the leading semiclassical order, meaning the first non-vanishing order in g at each topological level. The relevant equations which define these leading order instanton component fields are the self-duality equation (4.1) together with the fermion zero-mode equation

$$\bar{\mathcal{D}\!\!\!/}^{\dot{\alpha}\alpha} \lambda^A_\alpha = \ 0 \tag{4.2}$$

and the equation for the scalar field, which for the β-deformed theory takes the form

$$\mathcal{D}^2 \Phi^{AB} \ = \ \sqrt{2}\, i\, (\, \lambda^A * \lambda^B \ - \ \lambda^B * \lambda^A\,) \tag{4.3}$$

Here $\bar{\mathcal{D}\!\!\!/}^{\dot{\alpha}\alpha} = D^m \bar{\sigma}^{\dot{\alpha}\alpha}_m$ and $\mathcal{D}^2 = D^m D_m$ where D_m is the covariant derivative in the instanton background A_m.

Equation (4.1) specifies the gauge field instanton component A_m of topological charge k. The second equation (4.2) defines gaugino components λ^A of the instanton. As already mentioned, all $8kN$ adjoint fermion zero mode solutions of (4.2) are included in the k-instanton supermultiplet. Only 4 of these modes are protected by the $\mathcal{N} = 1$ superconformal invariance of the theory and are exact zero modes. Remaining $8kN - 4$ fermion zero modes will be lifted by interactions, this means that the instanton action will depend on collective coordinates of these fermion modes.

Finally, the last equation (4.3) determines the scalar field instanton components Φ^{AB} in terms of the gauge-field and gaugino components. β-deformation affects only this equation and it appears via the star product in the commutator on the right hand side. Apart from this obvious modification in (4.3), all three equations (4.1)–(4.3) are the same as in the undeformed $\mathcal{N} = 4$ SYM of ref. [8]. For convenience we have rescaled all the fields so that the only g dependence in the action is through the overall coefficient g^{-2}; the explicit g dependence in the Euler-Lagrange equations can be trivially restored by undoing this rescaling.

In order to evaluate correlators G_n in the SYM picture, one inserts the n appropriate gauge-invariant operators under the integration $\int d\mu^k_{\rm phys} \exp(-S^k_{\rm inst})$ where $S^k_{\rm inst}$ is the k-instanton action and $d\mu^k_{\rm phys}$ is the collective coordinate measure.

The ADHM gauge-field and the gaugino components of the instanton are parameterized by a set of collective coordinates. The scalar field is entirely determined in terms of A_m and λ^A in (4.3) and no new collective coordinates of the instanton are associated with Φ^{AB}. In general, there are $4kN$ *independent* bosonic and $8kN$ *independent* fermionic collective coordinates of a k-instanton configuration in our model. The simplest way to define the collective coordinate integration measure $d\mu^k_{\rm phys}$ is to consider an even larger set of instanton collective coordinates which are not all independent, but satisfy certain algebraic equations – the ADHM constraints (4.5).

These bosonic and fermionic collective coordinates live, respectively, in an $(N+2k)\times 2k$ complex matrix a, and in an $(N+2k) \times k$ Grassmann-valued complex matrix $\mathcal{M}^A$, where the $SU(4)_R$ index $A = 1, 2, 3, 4$ labels the supersymmetry. In components:[4]

$$a \;=\; \begin{pmatrix} w_{ui\dot\alpha} \\ (a'_{\beta\dot\alpha})_{li} \end{pmatrix}, \qquad \mathcal{M}^A \;=\; \begin{pmatrix} \mu^A_{ui} \\ (\mathcal{M}'^A_\beta)_{li} \end{pmatrix} \tag{4.4}$$

where both a'_m (defined by $a'_{\beta\dot\alpha} = a'_m \sigma^m_{\beta\dot\alpha}$) and $\mathcal{M}'^A_\beta$ are Hermitian $k \times k$ matrices. These matrices are subject to the ADHM constraints:

$$\mathrm{tr}_2\,(\tau^c \bar a a)_{ij} \;=\; 0\,, \qquad \left(\bar{\mathcal{M}}^A a + \bar a \mathcal{M}^A\right)_{\beta\, ij} \;=\; 0 \tag{4.5}$$

as well as to a $U(k)$ symmetry

$$w_{iu\dot\alpha} \to w_{ju\dot\alpha} U_{ji}\,, \quad a'_{mij} \to U^{-1}_{ik}\, a'_{mkl}\, U_{lj}\,, \quad U \in U(k)\,. \tag{4.6}$$

In the dilute instanton gas limit, the individual collective coordinates of the k far-separated instantons are

$$x^i_n \;=\; -(a'_n)_{ii}\,, \tag{4.7a}$$
$$\rho^2_i \;=\; \tfrac{1}{2}\bar w^{\dot\alpha}_{iu}\, w_{ui\dot\alpha}\,, \tag{4.7b}$$
$$(t^c_i)_{uv} \;=\; \rho_i^{-2}\, w_{ui\dot\alpha}\, (\tau^c)^{\dot\alpha}{}_{\dot\beta}\, \bar w^{\dot\beta}_{iv}\,, \tag{4.7c}$$

where x^i_n are positions, ρ_i are the sizes and t^c_i are the SU(2) generators of the individual instantons $i = 1, \ldots, k$.

The gauge field, gaugino and scalar field components of the k-instanton configuration which solve equations (4.1)–(4.3) can be found in ref. [8, 11] for the original $\mathcal{N} = 4$ SYM. The multi-instanton components in the β-deformed theory are obtained from the $\mathcal{N} = 4$ expressions in [8, 11] simply by applying the star products for all quantities charged under the $U(1) \times U(1)$ symmetry.

This multi-instanton configuration gives rise to the k-instanton action

$$S^k_{\rm inst} \;=\; \frac{8\pi^2 k}{g^2} + S^k_{\rm quad}\,. \tag{4.8}$$

Here $S^k_{\rm quad}$ is a term quadrilinear in fermionic collective coordinates, with one fermion collective coordinate chosen from each of the four gaugino sectors $A = 1, 2, 3, 4$:

$$S^k_{\rm quad} \;=\; \frac{\pi^2}{g^2}\,\epsilon_{ABCD}\,\mathrm{tr}_k\left(\Lambda_{A*B}\,\boldsymbol{L}^{-1}\,\Lambda_{C*D}\right). \tag{4.9}$$

The $k \times k$ anti-Hermitian fermion bilinear Λ_{A*B} is given by[5]

$$\Lambda_{A*B} \;:=\; \frac{1}{2\sqrt{2}}\left(\bar{\mathcal{M}}^A * \mathcal{M}^B - \bar{\mathcal{M}}^B * \mathcal{M}^A\right) \tag{4.10}$$

[4] We use notations of refs. [8, 11] throughout. The indices $u, v = 1, \ldots, N$ are $SU(N)$ indices; $\alpha, \dot\alpha$, etc. $=$ 1, 2 are Weyl indices (traced over with 'tr_2 '); $i, j = 1, \ldots, k$ (k being the topological number) are instanton indices (traced over with 'tr_k'); and $m, n = 1, 2, 3, 4$ are Euclidean Lorentz indices. Pauli matrices are $(\tau^c)_\alpha^{\ \beta}$ where $c = 1, 2, 3$.

[5] The $*$-subscript in Λ_{A*B} is used to indicate that the right hand side of (4.10) contains the star-product of the Grassmann collective coordinates $\bar{\mathcal{M}}^A$ and $\bar{\mathcal{M}}^B$.

and $\boldsymbol{L}$ is a linear self-adjoint operator that maps the k^2-dimensional space of such matrices onto itself:

$$\boldsymbol{L}\cdot\Omega \;=\; \tfrac{1}{2}\{\Omega,\bar{w}^{\dot{\alpha}}w_{\dot{\alpha}}\} + \; [a'_n,[\bar{a}'_n,\Omega]]\,. \tag{4.11}$$

The expression for the k-instanton action in the $\mathcal{N}=4$ theory was first derived in [42] and subsequently used in [8, 11]. The only modification of this expression in the β-deformed theory is the appearance of the star product between the fermionic collective coordinates in (4.10). In general, the Grassmann collective coordinates $\mathcal{M}^A$ and $\bar{\mathcal{M}}^A$ are so far the only parameters appearing in the instanton configuration which are charged under the $U(1)\times U(1)$. Hence the β-dependence is recovered from the $\mathcal{N}=4$ results by introducing the star products in expressions involving $\mathcal{M}^A$ and $\bar{\mathcal{M}}^A$ parameters.[6] In the original $\mathcal{N}=4$ theory, $S^k_{\rm quad}$ lifts all the adjoint fermion modes except the 16 exact supersymmetric and superconformal modes. The β-deformed theory lifts $8kN-4$ fermion modes. The unlifted modes are the two supersymmetric $\lambda^{\alpha\,A=4}_{\rm ss}$ and two superconformal modes $\lambda^{A=4}_{{\rm sc}\,\dot{\alpha}}$ of the unbroken $\mathcal{N}=1$ supersymmetry.

Following ref. [8], we now want to simplify $S^k_{\rm quad}$ by integrating in some new bosonic parameters. The idea is to replace the fermion quadrilinear $S^k_{\rm quad}$ with a fermion bilinear coupled to a set of auxiliary Gaussian variables. These take the form of an anti-symmetric tensor $\chi_{AB}=-\chi_{BA}$ whose elements are $k\times k$ matrices in instanton indices. We have

$$\begin{aligned}&\exp\left(-S^k_{\rm quad}\right) \;= \\ &\pi^{-3k^2}\left({\rm det}_{k^2}\boldsymbol{L}\right)^3\int d^{6k^2}\chi\,\exp\big[\,-{\rm tr}_k\left(\epsilon_{ABCD}\,\chi_{AB}\,\boldsymbol{L}\,\chi_{CD}\right)+4\pi i g^{-1}{\rm tr}_k\left(\chi_{AB}\,\Lambda_{A*B}\right)\,\big]\end{aligned} \tag{4.12}$$

The variable χ_{AB} transforms in the vector representation of the $SO(6)\cong {\rm SU}(4)$ R-symmetry and is subject to the reality condition $\chi^\dagger_{AB}=\frac{1}{2}\epsilon_{ABCD}\chi_{CD}$

Next we turn to the k-instanton $\mathcal{N}=4$ collective coordinate measure. This measure involves integrations over all bosonic and fermionic collective coordinates of the k-instanton, subject to the bosonic and fermionic ADHM constraints (4.5). These constraints are implemented via insertions of the appropriate delta functions [43]. The k-instanton integration measure for $\mathcal{N}=4$ SYM reads [8]:

$$\begin{aligned}\int d\mu^k_{\rm phys} \;&=\; \frac{2^{-k^2/2}(C_1)^k}{{\rm Vol}\,U(k)}\int d^{4k^2}a'\;d^{2kN}\bar{w}d^{2kN}w\prod_{A=1}^{4}d^{2k^2}\mathcal{M}'^A d^{kN}\bar{\mu}^A\;d^{kN}\mu^A({\rm det}_{k^2}\boldsymbol{L})^{-3}\\ &\times\prod_{r=1}^{k^2}\Big[\prod_{c=1,2,3}\delta\big(\tfrac{1}{2}{\rm tr}_k\,T^r({\rm tr}_2\;\tau^c\bar{a}a)\big)\prod_{A=1}^{4}\prod_{\dot{\alpha}=1,2}\delta\left({\rm tr}_k\,T^r(\bar{\mathcal{M}}^A a_{\dot{\alpha}}+\bar{a}_{\dot{\alpha}}\mathcal{M}^A)\right)\,.\end{aligned} \tag{4.13}$$

The constant C_1 is fixed at the 1-instanton level [8]

$$C_1 = 2^{-2N+1/2}\pi^{-6N}g^{4N} \tag{4.14}$$

[6]We note that Λ_{A*B} and $\epsilon_{ABCD}\,\Lambda_{C*D}$ are oppositely charged under the $U(1)\times U(1)$ and hence the star product is not needed on the right hand side of (4.9).

by comparing eq. (4.13) with the standard 1-instanton 't Hooft-Bernard measure [38, 44]. The integrals over the $k\times k$ matrices a'_n, $\mathcal{M}'^A$ and $\mathcal{A}^{AB}$ are defined in (4.13) as the integral over the components with respect to a Hermitian basis of $k\times k$ matrices T^r normalized so that $\mathrm{tr}_k\, T^r T^s = \delta^{rs}$.

The complete instanton partition function, $\int d\mu^k_{\rm phys}\exp(-S^k_{\rm inst})$, in the β-deformed gauge theory is given by[7]

$$
\begin{aligned}
&\frac{(C_1)^k 2^{-k^2/2}\pi^{-3k^2}}{\mathrm{Vol}\,U(k)}\int d^{4k^2}a'\, d^{2kN}\bar{w}\, d^{2kN}w\, d^{6k^2}\chi \prod_{A=1}^{4} d^{2k^2}\mathcal{M}'^A\, d^{kN}\bar{\mu}^A\, d^{kN}\mu^A \\
&\prod_{r=1}^{k^2}\left[\prod_{c=1,2,3}\delta\left(\frac{1}{2}\mathrm{tr}_k\, T^r(\mathrm{tr}_2\ \tau^c\bar{a}a)\right)\prod_{A=1}^{4}\prod_{\dot\alpha=1,2}\delta\left(\mathrm{tr}_k\, T^r(\bar{\mathcal{M}}^A a_{\dot\alpha}+\bar{a}_{\dot\alpha}\mathcal{M}^A)\right)\right] \\
&\exp\left[-\frac{8\pi^2}{g^2}-\mathrm{tr}_k\left(\epsilon_{ABCD}\,\chi_{AB}\,\boldsymbol{L}\,\chi_{CD}\right)+\frac{4\pi i}{g}\mathrm{tr}_k\left(\chi_{AB}\Lambda_{A*B}\right)\right]
\end{aligned}
\qquad (4.15)
$$

We note that this expression differs from the original $\mathcal{N}=4$ result of [8] only through the star product in the last term in the exponent. In the following section we will see that this fact has profound consequences.

In the appendix we also give an alternative string theory derivation of (4.15). We show there that our gauge theory expression (4.15) is identical to the partition function of k D-instantons in string theory with the β-deformation.

5. The large-N saddle-point integration: 1-instanton case

We will first carry out integrations over collective coordinates in the simplest case of a single instanton $k=1$. The generalization to multi-instantons will be discussed in the following section.

One way to carry out the single-instanton calculation, is to first solve the ADHM constraints (4.5), and then to integrate out the corresponding delta-functions in (4.15). The collective coordinates (4.4) which satisfy the $k=1$ ADHM constraints can be written in the simple canonical form [45, 6]:

$$
a = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \rho & 0 \\ 0 & \rho \\ \multicolumn{2}{c}{-x_0^m\sigma_m} \end{pmatrix}, \quad \mathcal{M}^A = \begin{pmatrix} \nu_1^A \\ \vdots \\ \nu_{N-2}^A \\ 4i\rho\bar{\eta}^{A1} \\ 4i\rho\bar{\eta}^{A2} \\ 4\xi_1^A \\ 4\xi_2^A \end{pmatrix} \qquad (5.1)
$$

Here we follow the usual notation where $\rho\in\mathbb{R}$ and $x_0^m\in\mathbb{R}^4$ denote the size and position of the instanton, and ξ^A_α and $\bar{\eta}^{A\dot\alpha}$ are the supersymmetric and superconformal fermion

[7] Note that (4.15) does not contain factors of $\det\boldsymbol{L}$; they cancelled out between eqs. (4.12) and (4.13).

zero modes, respectively. Equation (5.1) assumes the canonical 'North pole' embedding of the $SU(2)$ instanton within $SU(N)$; more generally there is a manifold of equivalent instantons obtained by acting on (5.1) by transformations Ω in the coset space $\Omega \in U(N)/(U(N-2) \times U(1))$. The complex Grassmann coordinates ν_i^A in eq. (5.1) (which do not carry a Weyl spinor index) may be thought of as the superpartners of the coset embedding parameters Ω. Together, ξ_α^A, $\bar\eta_{\dot\alpha}^A$, ν_i^A and $\bar\nu_i^A$ constitute $8N$ fermionic collective coordinates of a single instanton in the $\mathcal{N}=4$ SYM.

After integrating out the delta functions imposing the ADHM constraints, and integrating over the instanton iso-orientations Ω, the instanton measure (4.15) for $k=1$ reduces to:

$$\int d\mu^1_{\rm phys}\, e^{-S^1_{\rm inst}} = \frac{2^{-31}\pi^{-4N-5}g^{4N}}{(N-1)!(N-2)!}\int d^4x_0\, d\rho\, d^6\chi \prod_{A=1}^4 d^2\xi^A\, d^2\bar\eta^A\, d^{(N-2)}\nu^A\, d^{(N-2)}\bar\nu^A$$
$$\rho^{4N-7}\exp\left[-\frac{8\pi^2}{g^2} - 2\rho^2\chi^a\chi^a + \frac{4\pi i}{g}\,\chi_{AB}\Lambda_{A*B}\right]. \tag{5.2}$$

The integral is expressed in terms of the collective coordinates (5.1) and we have substituted the 1-instanton expression $\boldsymbol{L}=2\rho^2$. The fermion bilinear in the 1-instanton case reads

$$\Lambda_{A*B} = \frac{1}{2\sqrt{2}}\sum_{i=1}^{N-2}\left(e^{i\pi\beta_{AB}}\,\bar\nu_i^A\nu_i^B - e^{-i\pi\beta_{AB}}\,\bar\nu_i^B\nu_i^A\right) + i8\sqrt{2}\,\sin(\pi\beta_{AB})\,\left(\rho^2\bar\eta^A\cdot\bar\eta^B + \xi^A\cdot\xi^B\right) \tag{5.3}$$

In the expression above we used the fact the products of Grassmann parameters with the Weyl index $\xi^A\cdot\xi^B := \xi^{A\alpha}\xi^B_\alpha$ and $\bar\eta^A\cdot\bar\eta^B := \bar\eta^A_{\dot\alpha}\bar\eta^{B\dot\alpha}$ are symmetric in A and B. The 4×4 antisymmetric matrix β_{AB} was defined in (3.9).

It is worth noting that the second term on the right hand side of (5.3) is non-vanishing only for non-zero values of the deformation parameter β_{AB}. This implies that there are precisely four exact fermion zero modes in the β-deformed theory which do not enter (5.3): two supersymmetric ones, ξ^4_α, and two superconformal $\bar\eta^4_{\dot\alpha}$ modes. In the undeformed $\mathcal{N}=4$ theory, all 16 supersymmetric and superconformal modes were absent from Λ_{AB} and hence from the instanton action.

However, even though only 4 out of 16 supersymmetric and superconformal modes are exact, they will altogether be irrelevant for our purposes. The main point here is the fact that we can choose such correlators that all 16 of these modes will be saturated by the instanton expressions for the Yang-Mills operators inserted into the partition function (5.2). At the same time we require that all of the remaining ν and $\bar\nu$ modes in the instanton partition function should not be lifted by the insertions of the operators. For this to be correct, we first of all need to restrict ourselves to the insertions of gauge invariant composite operators which correspond to zero KK modes on the deformed sphere $\tilde S^5$ in supergravity.[8] Secondly, we have to restrict to the minimal correlators (2.19a)–(2.19d) of these operators. In the case of $\langle\Lambda^{16}\rangle$ correlator (2.19a) and the $\langle\mathcal{B}^8\rangle$ correlator (2.19b)

[8]Insertions of the operators corresponding to non-zero KK modes would lift some of the ν and $\bar\nu$ modes, as in the $\beta=0$ case studied in [8, 12].

all of the 16 supersymmetric/superconformal modes and none of the ν and $\bar{\nu}$ modes are lifted by the operator insertions.[9] In summary, for our purposes of calculating the minimal correlators in (2.19a), (2.19b) (and also in (2.19c), (2.19d) in the small-β regime) one can always neglect the second term on the right hand side of (5.3), which is what we will do from now on.

We can now start integrating out fermionic collective coordinates ν_i^A and $\bar{\nu}_i^A$ from the instanton partition function (5.2). For each value of $i = 1, \ldots, N-2$ this integration gives a factor of

$$\left(\frac{4\pi}{g}\frac{1}{\sqrt{2}}\right)^4 \det\nolimits_4 \left(e^{i\pi\beta_{AB}}\,\chi_{AB}\right) \tag{5.4}$$

The determinant above can be calculated directly. It will be useful to express the result in terms of the three complex variables X_i which are defined in terms of χ_{AB} in the way analogous to eqs. (3.15):

$$\begin{aligned} X_1 &= \chi^1 + i\chi^2 = 2\sqrt{2}\,\chi^\dagger_{23} = 2\sqrt{2}\,\chi_{14} \\ X_2 &= \chi^3 + i\chi^4 = 2\sqrt{2}\,\chi^\dagger_{31} = 2\sqrt{2}\,\chi_{24} \\ X_3 &= \chi^5 + i\chi^6 = 2\sqrt{2}\,\chi^\dagger_{12} = 2\sqrt{2}\,\chi_{34} \end{aligned} \tag{5.5}$$

In terms of these degrees of freedom, the β-deformed determinant takes the form

$$\begin{aligned} &\det\nolimits_4 \left(e^{i\pi\beta_{AB}}\,\chi_{AB}\right) = \\ &\frac{1}{64}\left((|X_1|^2+|X_2|^2+|X_3|^2)^2 \,-\, 4\,\sin^2(\pi\beta_R)\,(|X_1|^2|X_2|^2+|X_2|^2|X_3|^2+|X_1|^2|X_3|^2)\right) \end{aligned} \tag{5.6}$$

It follows that the determinant depends only on the three absolute values of $|X|$ and is independent of the three angles. We can further change variables as follows:

$$|X_i| = r\,\mu_i\,, \qquad \sum_{i=1}^{3}\mu_i^2 = 1 \tag{5.7}$$

and write

$$\left(\frac{4\pi}{g}\frac{1}{\sqrt{2}}\right)^4 \det\nolimits_4 \left(e^{i\pi\beta_{AB}}\,\chi_{AB}\right) = \left(\frac{\pi}{g}\right)^4 r^4\,\left(1\,-\,4\,\sin^2(\pi\beta_R)\,Q\right) \tag{5.8}$$

where

$$Q := \mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2\,. \tag{5.9}$$

[9] For the non-minimal correlators involving higher values of n this is not the case anymore. At the same time, even the minimal correlators $\langle\mathcal{E}^8\rangle$ and $\langle\mathcal{Q}^4\rangle$ in (2.19c)–(2.19d) can receive corrections from saturating some of the ν and $\bar{\nu}$ modes by the operator insertions. This would then require one to keep (part or all) of the 12 lifted supersymmetric/superconformal modes in the exponent. These corrections can in principle be straightforwardly calculated in the small β-limit, which is the regime relevant for comparison with the supergravity. We thank Stefano Kovacs for pointing this out to us. For more detail on the fermion-zero-mode structure of the operator insertions we refer the reader to ref. [12].

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We also split the integral over $d^6\chi$ in the partition function into an integral over r and the integral over the 5-sphere $\hat{\Omega}$:

$$\int d^6\chi = \int r^5\, dr\, d^5\hat{\Omega}\,, \qquad \text{where} \qquad \int d^5\hat{\Omega} \propto \int d\mu_1^2\, d\mu_2^2\, d\mu_3^2\; \delta\left(\sum_{i=1}^{3}\mu_i^2 - 1\right). \quad (5.10)$$

In summary after integrating out all of the ν and $\bar{\nu}$ fermionic collective coordinates we find

$$\begin{aligned}\int d\mu^1_{\rm phys}\, e^{-S^1_{\rm inst}} \;&=\; \frac{g^8}{2^{31}\pi^{13}(N-1)!(N-2)!}\int d^4x_0\, \frac{d\rho}{\rho^5}\, d^5\hat{\Omega} \prod_{A=1,2,3,4} d^2\xi^A\, d^2\bar{\eta}^A \\ &\qquad e^{-\frac{8\pi^2}{g^2}}\, \left(1\,-\,4\,\sin^2(\pi\beta_R)\,Q\right)^{N-2}\; \rho^{4N-2}\, I_N \quad (5.11)\end{aligned}$$

Here I_N denotes the r integration, which it is instructive to separate out:

$$I_N = \int_0^\infty dr\, r^{4N-3}\, e^{-2\rho^2 r^2} = \frac{1}{2}(2\rho^2)^{-2N+1}\int_0^\infty dx\, x^{2N-2}\, e^{-x} = \frac{1}{2}(2\rho^2)^{1-2N}(2N-2)! \quad (5.12)$$

From eqs. (5.11)–(5.12) one sees that the x_0 and ρ integrations assemble into the scale-invariant AdS_5 volume form $d^4x_0\, d\rho\, \rho^{-5}$ and also the integration over the 5-sphere arises from $d^5\hat{\Omega}$. However, it also follows that the final result given by eqs. (5.11)–(5.12) is so far unsatisfactory from the perspective of the supergravity interpretation. First of all, the χ variables gave rise to the integration over the *undeformed* sphere S^5. Secondly, the β_R-dependent factor, $\sin^2(\pi\beta_R)\,Q$, is neither spherically-invariant (due to its Q-dependence) nor can it be easily associated with the deformed sphere $\tilde{S}^5$. Finally, the exponent of the instanton action in (5.12) is of the form $e^{\frac{-8\pi^2}{g^2+i\theta}} = e^{2\pi i\tau_0}$ which is not of the form $e^{2\pi i\tau}$ expected in supergravity.

Quite remarkably, all of these perceived problems of eqs. (5.11)–(5.12) can be resolved by taking the large-N limit and carefully specifying the appropriate order of limits procedure. We will now describe this procedure in detail.

On the gauge theory side we have no choice but to work in the weak-coupling limit. To justify working with the leading-order instanton and neglecting and infinite set of higher order terms in perturbation theory in the instanton background, we must take the limit $g^2N \to 0$ and only after that impose the large N limit. In addition, so far we have been treating the β-deformation parameter as an independent fixed constant. However, as we have seen in section 2, the validity of the Lunin–Maldacena supergravity solution is restricted to the regime of small β. We proceed with the Yang-Mills instanton calculation by taking the limits in the following order:

1. $g^2 \ll 1$ with N and β_R being fixed
2. $N \gg 1\,,\quad \beta_R^2 \ll 1$

The second limit shall be taken in such a way that $\beta_R^2 N$ is held fixed. In fact, our Yang-Mills results are more general than that and hold for any values of the parameter $\beta_R^2 N$. It can be shown that the derivation below holds for $\beta_R^2 N \gg 1$ and $\beta_R^2 N \ll 1$.

We now consider the factor

$$\mathcal{F} := e^{-\frac{8\pi^2}{g^2}}\,\left(1\,-\,4\,\sin^2(\pi\beta_R)\,Q\right)^{N-2} \tag{5.13}$$

which appears on the right hand side of (5.11) and simplify it according to the limits above. First, we replace the $\sin^2(\pi\beta_R)$ by $(\pi\beta_R)^2$ which is justified since $\beta_R^2 \ll 1$. Next, we note that in our limit

$$\left(1\,-\,4\,(\pi\beta_R)^2\,Q\right)^{N-2}\,\sim\,\exp\left[N\,\log(1-4\pi^2\beta_R^2\,Q)\right]\,\sim\,\exp\left[-N\beta_R^2\,4\pi^2\,Q\right]. \tag{5.14}$$

In the last expression above we have neglected the higher-order terms $\mathcal{O}(N\beta_R^4 Q^2) \sim 1/N \ll 1$ which arose from the expansion of the logarithm.

This allows us to write down the $\mathcal{F}$-factor defined in (5.13) as

$$\mathcal{F}\,=\,\exp\left[-\frac{8\pi^2}{g^2}\,-\,N\beta_R^2\,4\pi^2\,Q\right]\,=\,\exp\left[-\frac{8\pi^2}{g^2}\left(1\,+\,\tfrac{1}{2}\,N\beta_R^2 g^2\,Q\right)\right]. \tag{5.15}$$

We stress that the expression above is exact in the ordered weak-coupling–large-N–small-β limit which we use in our semi-classical Yang-Mills calculation.

We now compare this $\mathcal{F}$-factor arising from the Yang-Mills instanton in the weak coupling limit to $e^{2\pi i\tau}$ in the Lunin-Maldacena supergravity solution. We recall that τ is the parameter which combines the natural dilaton and the axion of the deformed supergravity solution

$$\tau\,=\,ie^{-\phi}\,+\,C\,:=\,i\tau_2\,+\,\tau_1 \tag{5.16}$$

This τ is related to τ_0 and hence to the SYM couplings via a non-trivial function G as follows[10]

$$e^{-2\pi\tau_2}\,=\,e^{-2\pi\tau_{02}\,G^{-1/2}}\,,\qquad \tau_0\,=\,\frac{2\pi i}{g^2}+\frac{\theta}{2\pi}\,:=\,i\tau_{02}\,+\,\tau_{01} \tag{5.17}$$

Here G is the function of the coordinates μ_i on the deformed sphere, and it also depends on the deformation parameter $\hat\gamma$

$$G^{-1}\,=\,1\,+\,\hat\gamma^2(\mu_1^2\mu_2^2+\mu_2^2\mu_3^2+\mu_1^2\mu_3^2)\,,\qquad \hat\gamma^2\,=\,\beta_R^2 N g^2 \tag{5.18}$$

We now want to take the same weak-coupling–large-N–small-β limit of the supergravity expressions (5.17)–(5.18). We find

$$\begin{aligned} e^{-2\pi\tau_2}\,&=\,\exp\left[-\frac{8\pi^2}{g^2}\left(1\,+\,N\beta_R^2 g^2\,Q\right)^{\frac{1}{2}}\right] \\ &=\,\exp\left[-\frac{8\pi^2}{g^2}\,-\,4\pi^2\,N\beta_R^2\,Q+\,\sim\,(N\beta_R^2)^2 g^2 Q^2\,+\,\ldots\right]. \end{aligned} \tag{5.19}$$

The linear in Q term is unaffected in the limit, while the quadratic term is negligible, $(N\beta_R^2)^2 g^2 Q^2\,\sim\,g^2\,\ll 1$. This gives

$$e^{2\pi i\tau}\,=\,\mathcal{F} \tag{5.20}$$

where we have restored the θ parameter in the Yang-Mills instanton action.

[10]The axion C or the Yang-Mills θ parameter are not changed by this transformation and the real parts of the two τ's are the same $\tau_1\,=\,\tau_{01}$. When working with instantons we will not pay attention to the θ parameter, if required it can always be trivially restored in the instanton action.

From these considerations we conclude that the $\mathcal{F}$-factor which arises from the Yang-Mills instanton calculation in the semi-classical limit of the deformed theory is equivalent to the corresponding supergravity factor $e^{2\pi i\tau}$ of eq. (5.17) in the Lunin-Maldacena background. For this equivalence it is necessary to identify the μ_i coordinates of the instanton χ-collective-coordinates defined in (5.7), (5.9) with the μ_i coordinates of the deformed $\tilde{S}^5$ sphere of the Lunin-Maldacena background. This implies that the integration measure over the 'angles of' χ_a, or more precisely over the the 5-dimensional manifold $\hat{\Omega}$ in (5.10) should correspond to the volume element on the deformed $\tilde{S}^5$ sphere. This volume element $\omega_{\tilde{S}^5}$ over the deformed sphere $\tilde{S}^5$ can be found from the Lunin-Maldacena metric. In the string frame we get

$$\int \omega_{\tilde{S}^5} = \int \omega_{S^5} G \tag{5.21}$$

where ω_{S^5} is the volume element of the original 5-sphere, and G is given in (5.18).

We have seen above that when $G^{-1/2}$ appears in the exponent weighted with the instanton action, it gives rise to two terms in the semiclassical limit: the order-1 term and the order-Q term. However, when G appears in the pre-exponent, as on the right hand side of (5.21), it is indistinguishable from unity. Indeed,

$$G = (1 + g^2 N\beta_R Q)^{-1} = 1 - g^2 N\beta_R^2 Q + \ldots \rightarrow 1 \tag{5.22}$$

since $g^2 N\beta_R^2 Q \sim g^2 \ll 1$ and should be neglected. This amounts to identifying

$$\int \omega_{\tilde{S}^5} = \int d^5\hat{\Omega}\, G \rightarrow \int d^5\hat{\Omega} \tag{5.23}$$

For consistency we can also re-calculate I_N in (5.11) in the large-N limit. First one rescales $r \rightarrow \sqrt{N} r$, or equivalently, $\chi^a \rightarrow \sqrt{N}\chi^a$, so that N factors out of the exponent. The integral then becomes

$$I_N = N^{2N-1} \int_0^\infty dr\, r^{-3}\, e^{2N(\log r^2 - \rho^2 r^2)}\,, \tag{5.24}$$

which is in a form amenable to a standard saddle-point evaluation. The saddle-point is at $r = \rho^{-1}$ and, to leading order, a Gaussian integral around the solution gives

$$\lim_{N\rightarrow\infty} I_N = \rho^{2-4N} N^{2N-1} e^{-2N} \sqrt{\frac{\pi}{N}}\,, \tag{5.25}$$

which is valid up to $1/N$ corrections.

Our final result for the single-instanton partition function in the semi-classical large-N limit takes the following simple form:

$$\begin{aligned}\int d\mu^1_{\rm phys}\, e^{-S^1_{\rm inst}} &\rightarrow \frac{\sqrt{N} g^8}{2^{33}\pi^{27/2}} \int \frac{d^4x_0\, d\rho}{\rho^5}\, d^5\hat{\Omega} \prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A\; e^{-2\pi\tau_{02}G^{-1/2}+2\pi\tau_1} \\ &= \frac{\sqrt{N} g^8}{2^{33}\pi^{27/2}} \int d^{10}X \sqrt{-g_{10}} \prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A\; e^{2\pi i\tau} \end{aligned} \tag{5.26}$$

where the integration over $d^{10}X\,\sqrt{-g_{10}}$ is the integration of the 10-dimensional space which corresponds to the Lunin-Maldacena background. This 10-dimensional bosonic integration can be factored into the AdS_5-part parameterized by the instanton position and the scale-size, $d^4x_0\,\frac{d\rho}{\rho^5}$, times the 5-dimensional integration over the deformed sphere $\tilde{S}^5$ parameterized by $d^5\hat{\Omega}\,G\;\to\;d^5\hat{\Omega}$. The main feature of our 1-instanton result (5.26) is the appearance of the complete dilaton-axion factor in the exponent, $e^{2\pi i\tau}$.

6. Multi-instanton large-N integration

In this section we return to the general case of k instantons. Following the saddle-point approach of [8] and also building upon the 1-instanton calculation of the previous section, we will evaluate the multi-instanton partition function (4.15).

The first step is to reduce the k-instanton measure to the $SU(N)$-gauge-invariant expression [8]. The expression (4.15) can be simplified by transforming to a smaller set of gauge-invariant collective coordinates (i.e., variables without an uncontracted 'u' index). In the bosonic sector this means changing variables from $\{w,\bar{w}\}$ to the W variables introduced as follows:

$$\left(W^{\dot{\alpha}}_{\ \dot{\beta}}\right)_{ij}=\bar{w}^{\dot{\alpha}}_{iu}\,w_{ju\dot{\beta}}\,,\quad W^0=\mathrm{tr}_2\,(W)\,,\quad W^c=\mathrm{tr}_2\,(\tau^cW)\,,\quad c=1,2,3\,. \tag{6.1}$$

This enables us to reduce the number of bosonic integrations using the Jacobian identity is proved in [8]:

$$d^{2Nk}\bar{w}\,d^{2Nk}w\;=\;c_{k,N}\,\left(\mathrm{det}_{2k}W\right)^{N-2k}\,d^{k^2}W^0\prod_{c=1,2,3}d^{k^2}W^c\,, \tag{6.2}$$

where $c_{k,N}$ is a constant. An important feature of this change of variables is that it allows us to eliminate the bosonic ADHM constraints [43, 8]. The ADHM constraints in eq. (4.15), which are quadratic in the $\{w,\bar{w}\}$ coordinates, become linear in terms of W

$$0\;=\;W^c\;+\;[\,a'_n\,,\,a'_m\,]\,\mathrm{tr}_2\,(\tau^c\bar{\sigma}^{nm})\;=\;W^c\;-\;i\,[\,a'_n\,,\,a'_m\,]\,\bar{\eta}^c_{nm}\,. \tag{6.3}$$

We therefore use eq. (6.3) to eliminate W^c from the measure together with the delta-functions of the bosonic ADHM constraints. Furthermore, we note that as $N\to\infty$, the Jacobian factor of $(\mathrm{det}W)^N=\exp(N\,\mathrm{tr}\log W)$ in (6.2) will be amenable to a saddle-point treatment.

In the fermion sector, following [43, 8], we change variables from $\{\mu,\bar{\mu}\}$ to $\{\zeta,\bar{\zeta},\nu,\bar{\nu}\}$ defined by

$$\mu^A_{iu}=w_{uj\dot{\alpha}}\,\zeta^{\dot{\alpha}A}_{ji}+\nu^A_{iu}\,,\qquad\bar{\mu}^A_{iu}=\bar{\zeta}^A_{\dot{\alpha}ij}\,\bar{w}^{\dot{\alpha}}_{ju}+\bar{\nu}^A_{iu}\,, \tag{6.4}$$

where the ν modes form a basis for the $\perp$-space of w:

$$0\;=\;\bar{w}^{\dot{\alpha}}_{iu}\nu^A_{ju}\;=\;\bar{\nu}^A_{iu}w_{ju\dot{\alpha}}\,, \tag{6.5}$$

In these variables the fermionic ADHM constraints in eq. (4.15) have the gauge-invariant form

$$0\;=\;\bar{\zeta}^A\,W+W\,\zeta^A+[\mathcal{M}'^A,a'] \tag{6.6}$$

which can be used to eliminate $\bar{\zeta}^A$ in favor of ζ^A and $\mathcal{M}'^A$; doing so gives a factor which precisely cancels the Jacobian for the change of variables (6.4).

Since the ν and $\bar{\nu}$ modes are absent from the constraint (6.6), they can now be straightforwardly integrated out. First, we decompose $\Lambda_{A*B} = \hat{\Lambda}_{A*B} + \tilde{\Lambda}_{A*B}$, where

$$(\hat{\Lambda}_{A*B})_{ij} = \frac{1}{2\sqrt{2}} \left(\bar{\nu}^A_{iu} * \nu^B_{ju} - \bar{\nu}^B_{iu} * \nu^A_{ju} \right) \,, \tag{6.7}$$

and

$$\tilde{\Lambda}_{A*B} = \frac{1}{2\sqrt{2}} \left(\, \bar{\zeta}^A * W\zeta^B - \bar{\zeta}^B * W\zeta^A + [\mathcal{M}'^A \,,\, \mathcal{M}'^B]_* \, \right) . \tag{6.8}$$

Second, we calculate

$$\int d^{4k(N-2k)}\nu \, d^{4k(N-2k)}\bar{\nu} \, \exp\left(\frac{4\pi i}{g} \, \mathrm{tr}_k(\chi_{AB}\hat{\Lambda}_{A*B}) \right) = \\ \left(\frac{8\pi^2}{g^2} \right)^{2k(N-2k)} \left(\det{}_{4k} \, e^{i\pi\beta_{AB}}\chi_{AB} \right)^{N-2k} \tag{6.9}$$

To simplify notation we introduce a notation $q_{AB} = e^{i\pi\beta_{AB}}$, so the determinant in (6.9) can be written as[11] $(\det_{4k} q\chi)^N$. It too will contribute to the saddle-point equations in the large-N limit, similarly to the $(\det W)^N$ factor in eq. (6.2). The third and final contribution to these equations will be the Gaussian term $\chi \boldsymbol{L} \chi$ in eq. (4.15), once one rescales $\chi_{AB} \to \sqrt{N}\chi_{AB}$ so that N factors out in front. Combining the above manipulations, we write down the final expression for the $SU(N)$-gauge-invariant measure (cf. [8]):

$$\int d\mu^k_{\mathrm{phys}} \, e^{-S^k_{\mathrm{inst}}} \;= \\ \frac{g^{8k^2} N^{k^2} e^{-8\pi^2 k/g^2}}{2^{27k^2/2-k/2} \, \pi^{13k^2} \, \mathrm{Vol}(U(k))} \int d^{k^2}W^0 \, d^{4k^2}a' \, d^{6k^2}\chi \prod_{A=1,2,3,4} d^{2k^2}\mathcal{M}'^A \, d^{2k^2}\zeta^A \tag{6.10}$$

$$\times \; (\det{}_{2k}W \det{}_{4k} q\chi)^{-2k} \, \exp\left[- \; N S^{k\,(\beta)}_{\mathrm{eff}} \; + \; 4\pi i g^{-1}\sqrt{N} \, \mathrm{tr}_k(\chi_{AB}\tilde{\Lambda}_{A*B}) \right] \tag{6.11}$$

The constant in front of the integral is written in the large-N limit. In the exponent in the last line of (6.11) we have grouped all of the order-N terms into the quantity $N S^{k\,(\beta)}_{\mathrm{eff}}$. The quantity $S^{k\,(\beta)}_{\mathrm{eff}}$ is the sum of the three terms relevant for the large-N saddle point approach mentioned above plus a constant piece

$$S^{k\,(\beta)}_{\mathrm{eff}} \;:=\; -\mathrm{tr}_{2k}\log W \;-\; \mathrm{tr}_{4k}\log q\chi \;+\; \epsilon_{ABCD}\, \mathrm{tr}_k\left(\chi_{AB}\, \boldsymbol{L}\, \chi_{CD}\right) \;-\; 2k\,(1 + 3\log 2) \,. \tag{6.12}$$

This expression involves the $11k^2$ bosonic variables comprising the eleven independent $k \times k$ Hermitian matrices W^0, a'_n and χ_a. As mentioned earlier, the remaining components W^c, $c = 1, 2, 3$, are eliminated in favor of the a'_n via the ADHM constraint. The action is also invariant under the $U(k)$ symmetry which acts by adjoint action on all the variables.

We can apply the large-N saddle-point formalism to the integral in (6.11), but before doing so we want to slightly simplify $S^{k\,(0)}_{\mathrm{eff}}$ with respect to its β_R-dependence. Specifically, we consider the $\mathrm{tr}_{4k}\log q\chi$ term in $S^{k\,(\beta)}_{\mathrm{eff}}$ and split the $U(k)$ variables χ_{ij} into the sum of

[11] Note, that $q\chi$ is a shorthand for $q_{AB}\chi_{AB}$ which is a product of matrix elements and not not the product of two matrices.

the $U(1)$ variables $\chi^\star \delta_{ij}$ and the $SU(k)$ degrees of freedom $\hat{\chi}_{ij}$

$$\chi_{Ai\,Bj} \;=\; \chi^\star_{AB}\delta_{ij} \;+\; \hat{\chi}_{Ai\,Bj}\,, \qquad \mathrm{tr}_k\,\hat{\chi}_{AB} \;=\; 0\,. \tag{6.13}$$

We then expand the $\mathrm{tr}_{4k}\log q\chi$ as follows

$$S^{k\,(\beta)}_{\text{eff}} \;\in\; N\,\mathrm{tr}_{4k}\log q\chi \;=\; Nk\,\mathrm{tr}_4\log q\chi^\star \;+\; N\,\mathrm{tr}_{4k}\log\,\left(1+(q\chi^\star)^{-1}(q\hat{\chi})\right) \tag{6.14}$$

and further re-write it as

$$Nk\,\mathrm{tr}_4(\log q\chi^\star \;-\; \log\chi^\star) \;+\; N\,\mathrm{tr}_{4k}\log\,\left(\chi^\star+\chi^\star(q\chi^\star)^{-1}(q\hat{\chi})\right). \tag{6.15}$$

We can now take the small-β_R limit (accompanied by the large-N limit). The first term on the right hand side in (6.15) is equal to k times the single instanton result derived in the previous section, which is k times $N\beta_R^2\,4\pi^4\,Q$. Hence, for this term, the relevant contribution comes at the order-$N\beta_R^2$ in the $N\to\infty$, $\beta_R\to 0$ limit.

However, a careful fluctuations analysis along the lines of [8], shows that in the second term in (6.15) the dominant contribution comes at the order $N(\beta_R)^0 \sim N$. This makes all higher-order terms in β_R suppressed in the limit. Thus we set $q=1$ in the second term which then reads:

$$N\,\mathrm{tr}_{4k}\log\,(\chi^\star+\hat{\chi}) \;=\; N\,\mathrm{tr}_{4k}\log\chi \tag{6.16}$$

In summary, we express

$$NS^{k\,(\beta)}_{\text{eff}} \;=\; -\,Nk\,\mathrm{tr}_4(\log q\chi^\star \;-\; \log\chi^\star) \;+\; S^{k\,(0)}_{\text{eff}} \tag{6.17}$$

where $S^{k\,(0)}_{\text{eff}}$ is given by (6.12) with the substitution $\beta=0$ or equivalently $q=1$. The first term in (6.17) is combined with the k-instanton action $8\pi^2k/g^2$ in exactly the same way as in eqs. (5.15), (5.26) in the previous section. This amounts to promoting the Yang-Mills multi-instanton gauge action to the appropriate τ-dependence required in the supergravity effective action

$$\exp\left[-\frac{8\pi^2k}{g^2}+ik\theta \;-\; Nk\beta_R^2\,4\pi^2\,Q\right] \;=\; e^{-2\pi k\tau_{02}G^{-1/2}+2\pi k\tau_1} \;=\; e^{2\pi ik\tau} \tag{6.18}$$

What remains is $S^{k\,(0)}_{\text{eff}}$, which does not depend on the deformation parameter, it is the same as in the $\mathcal{N}=4$ SYM theory and is amendable to the large-N saddle-point treatment. The saddle-point approach has been set up and the integrations around the saddle-point solution have been carried out in [8]. Here we will only give a brief summary of the result. It turns out that the dominant contribution contribution to the integral comes from the maximally degenerate saddle-point solution:

$$W^0 = 2\rho^2\,1_{[k]\times[k]}, \qquad \chi_a = \rho^{-1}\hat{\Omega}_a\,1_{[k]\times[k]}, \qquad a'_n = -x_n\,1_{[k]\times[k]}\,, \tag{6.19}$$

which corresponds to k coincident Yang-Mills instantons of the same scale-size ρ which live in the mutually commuting $SU(2)$ subgroups of the $SU(N)$. In the supergravity interpretation this saddle-point corresponds to a configuration living at a common point $\{x_n,\hat{\Omega}_a,\rho\}$ in the deformed $AdS_5\times\tilde{S}^5$. This is a point-like object – the D-instanton of charge k.

Around the special solution, the bosonic fluctuations fall into three sets. First, there are 10 zero modes which correspond to the position of the k-instanton "bound state" in $AdS_5 \times \tilde{S}^5$. These are exactly the same as in the 1-instanton case. Second, there are k^2 fluctuations called φ which have a nonzero quadratic coefficient in the small-fluctuations expansion. The remaining $10k^2 - 10$ fluctuations first appear beyond quadratic order and they correspond to the traceless i.e. $SU(k)$ parts $\hat{\chi}_{AB}$, and $\hat{a}'_m$ of the ten $k \times k$ matrices χ_{AB} and a'_m. Since fluctuations over φ are Gaussian, they can be straightforwardly integrated out.

To complete the expansion, we include the fermion terms in the exponent of (6.11). The second term in the exponent involves fermionic degrees of freedom appearing in $\tilde{\Lambda}_{A*B}$. Here we again are interested in the leading order non-vanishing contributions in the $\beta_R \to 0$ limit. This amounts to dropping the star product $\tilde{\Lambda}_{A*B} \to \tilde{\Lambda}_{AB}$. The resulting fermion terms in the exponent involve the traceless parts $\hat{\zeta}^{\dot{\alpha}A}$ and $\hat{\mathcal{M}}'^A_\alpha$ coupled to $\hat{a}'_m$ and $\hat{\chi}_{AB}$.

Remarkably, in the large-N limit, the leading-order terms of the effective action around the saddle-point solution, with the quadratic fluctuations φ integrated out, precisely assemble themselves into the dimensional reduction from ten to zero of $\mathcal{N} = 1$ supersymmetric Yang-Mills with gauge group SU(k) in *flat* space. The SU(k) adjoint-valued ten-dimensional gauge field and Majorana-Weyl fermion are defined in terms of the fluctuations:

$$A_\mu = N^{1/4}\left(\rho^{-1}\hat{a}'_m, \rho\hat{\chi}^a\right), \qquad \Psi = \left(\frac{\pi}{2g}\right)^{1/2} N^{1/8}\left(\rho^{-1/2}\hat{\mathcal{M}}'^A_\alpha, \rho^{1/2}\hat{\zeta}^{\dot{\alpha}A}\right). \tag{6.20}$$

The action for the dimensionally reduced gauge theory is

$$S(A_\mu, \Psi) = -\frac{1}{2}\mathrm{tr}_k\,[A_\mu, A_\nu]^2 \;+\; \mathrm{tr}_k\left(\bar{\Psi}\Gamma_\mu\,[A_\mu, \Psi]\right). \tag{6.21}$$

We conclude that the effective gauge-invariant measure for k instantons in the large-N limit factorizes into a 1-instanton-like piece, for the position of the bound state in $AdS_5 \times S^5$ and the 16 supersymmetric and superconformal modes, times the partition function $\mathcal{Z}_k$ of the dimensionally-reduced $\mathcal{N} = 1$ supersymmetric SU(k) gauge theory in flat space:

$$\begin{aligned}\int d\mu^k_{\mathrm{phys}}\, e^{-S^k_{\mathrm{inst}}} \;&=\; \frac{\sqrt{N}g^8}{k^3 2^{17k^2/2-k/2+25}\,\pi^{9k^2/2+9}} \\ &\quad\times \int \frac{d\rho}{\rho^5}\, d^4x\, d\Omega_5 \prod_{A=1,2,3,4} d^2\xi^A d^2\bar{\eta}^A\, e^{-8\pi^2 k/g^2}\, \hat{\mathcal{Z}}_k\,,\end{aligned} \tag{6.22}$$

where $\hat{\mathcal{Z}}_k$ is the partition function of an $\mathcal{N} = 1$ supersymmetric SU(k) gauge theory in ten dimensions dimensionally reduced to zero dimensions:

$$\hat{\mathcal{Z}}_k \;=\; \frac{1}{\mathrm{Vol}\,SU(k)} \int_{SU(k)} d^{10}A\, d^{16}\Psi\, e^{-S(A_\mu,\Psi)}\,. \tag{6.23}$$

Notice that the rest of the measure, up to numerical factors, is independent of the instanton number k. When integrating expressions which are independent of the SU(k) degrees-of-freedom, $\hat{\mathcal{Z}}_k$ is simply an overall constant factor. A calculation of ref. [46, 47] revealed that $\hat{\mathcal{Z}}_k$ is proportional to $\sum_{d|k} d^{-2}$, a sum over the positive integer divisors d of k. In our

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notation we have [46, 47, 8]:

$$\hat{\mathcal{Z}}_k = 2^{17k^2/2-k/2-8}\pi^{9k^2/2-9/2}k^{-1/2}\sum_{d|k}\frac{1}{d^2}\,. \tag{6.24}$$

In summary, on gauge invariant and SU(k) singlet operators, our effective large-N collective coordinate measure has the following simple form:

$$\int d\mu^k_{\rm phys}\, e^{-S^k_{\rm inst}} = \frac{\sqrt{Ng^2}}{2^{33}\pi^{27/2}}\,\frac{k}{g^2}^{-7/2}\sum_{d|k}\frac{1}{d^2}\int\frac{d^4x\,d\rho}{\rho^5}\,d^5\hat{\Omega}\prod_{A=1,2,3,4}d^2\xi^A d^2\bar{\eta}^A e^{2\pi ik\tau}\,. \tag{6.25}$$

We can already identify a number of key features of the k-instanton measure which are important for the comparison with the supergravity results (2.14)–(2.18). First, the factor of $(k/g^2)^{-7/2}$ in the measure maps nicely to the factor in $(k/g^2)^{n-7/2}$ for $n = 0$ on the right hand side of (2.18). Second, we recognise the inverse divisors squared contributions $\sum_{d|k}\frac{1}{d^2}$ in (6.25) and (2.18). The matching of $e^{2\pi ik\tau}$ factors has been mentioned earlier and it is one of our main results. The factor of $\sqrt{Ng^2}$ in (6.25) gives rise to $(\alpha')^{-1}$ in (2.14), and the volume element of the AdS_5 is represented via $d^4x\,d\rho\,/\rho^5$ in (6.25). Finally, the integration over $d^5\hat{\Omega}$ gives rise to the volume factor of the 5-sphere. However, as we have already explained earlier, in our semi-classical limit we cannot distinguish between the deformed and the undeformed spheres *in the pre-exponent*. The deformation is, however, manifest in the exponential factor $e^{2\pi ik\tau}$.

7. Correlation functions

Finally, we can use our measure to calculate the correlation functions $G_n(x_1,\ldots,x_n)$ listed in (2.19a)–(2.19d). This entails inserting into eq. (6.25) the appropriate product of gauge-invariant composite chiral operators $\mathcal{O}_1(x_1)\times\cdots\times\mathcal{O}_n(x_n)$, which together contain the requisite 16 exact fermion modes to saturate the 16 Grassmann integrations[12] in (6.25). Since, at leading order in N, the k instantons sit at the same point in $AdS_5\times\tilde{S}^5$, it follows that $\mathcal{O}^{(k)}_j$ is simply proportional to its single-instanton counterpart: $\mathcal{O}^{(k)}_j = k\mathcal{O}^{(1)}_j$. Therefore G_n scales like $(k/g^2)^n$. This promotes the factor in the partition function to the full value required to match with (2.18)

$$\left(\frac{k}{g^2}\right)^{-7/2} \longrightarrow \left(\frac{k}{g^2}\right)^{n-7/2} \tag{7.1}$$

and, as before, factors of G in the pre-exponent in (2.18) cannot be tested in our limit.

Furthermore, it was shown in [5, 8] that the instanton contributions to the operators $\mathcal{O}$ precisely match the functional form of the bulk-to-boundary propagators in (2.20). We thus conclude that our Yang-Mills multi-instanton results for the correlators G_n which follow from eq. (6.25) completely reconstruct the supergravity expressions eqs. (2.14)–(2.18), (2.20).

[12] We also refer the reader to the earlier discussion following eq. (5.3) and to footnote 9.

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As the final comment we recall that the matching between the supergravity and the SYM results holds in the opposite limits. The SYM expression is derived in the weak coupling limit $g^2 N \to 0$, $N \to \infty$ while the supergravity is a good approximation to string theory in the strong coupling limit $g^2 N \to \infty$, $N \to \infty$. This use of different limits on the two sides of the AdS/CFT correspondence is, of course, the consequence of the strong-to-weak coupling nature of the AdS/CFT. Nevertheless, even though the two sets of limits are mutually exclusive, we have shown that the leading order results in the SYM and in supergravity agree with each other. This agreement between the strong and the weak coupling limits holds in the instanton case (as it did hold in the original $\mathcal{N} = 4$ settings in [5−8]), but it is not expected to hold in perturbation theory. At present no non-renormalization theorem is known which would apply to these instanton effects and explain the agreement. However the fact that there is an agreement between the results on the two sides of the correspondence must imply a non-trivial consistency of the AdS/CFT. We refer the reader to refs. [8, 48] for a more detailed discussion on this point.

8. Generalization to complex β deformations

We conclude with a brief generalization of our instanton approach to the general case of complex β deformations. These are defined on the SYM side by the superpotential

$$ih \, \mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) \,. \tag{8.1}$$

Here h and β are two complex parameters which have to satisfy the Leigh-Strassler constraint [33] in order to ensure that the resulting theory is conformally invariant at the quantum level. Anticipating taking the small β limit and also working to the leading order in perturbation theory around the instanton, the Leigh-Strassler constraint simplifies to the condition $h = g$.

The instanton configuration at the leading order in g is still defined by equations (4.1)–(4.3), with the scalar field equation (4.3) taking the form

$$\mathcal{D}^2\Phi^{AB} \;=\; \sqrt{2}\, i \,(\, e^{i\pi\beta_{AB}}\, \lambda^A\lambda^B \;-\; e^{-i\pi\beta_{AB}}\, \lambda^B\lambda^A \,) \tag{8.2}$$

where we have set $h = g$. The β parameter is complex, $\beta_{AB} := \beta_{R\,AB} - i\sigma_{AB}$ and furthermore is assumed to be small. The only modification of our approach in previous sections required for complex values of β is the modification of the β-deformed determinant in eq. (5.4) which arises from the integration of the ν and $\bar{\nu}$ fermion zero modes in the one-instanton background. We now need to calculate the complex-β-deformed counterpart of the determinant of the equation (5.6) with the complex parameter $\beta = \beta_R - i\sigma$ and in the $|\beta|^2 \ll 1$ limit. By following the same steps which lead from (5.6) to (5.15) we find

$$\mathcal{F} = \exp\left[-\frac{8\pi^2}{g^2} + i\theta \,-\, N(\beta_R^2 - \sigma^2 - 2i\beta_R\sigma)\, 4\pi^2\, Q \right] . \tag{8.3}$$

In what follows we will need the expressions for the dilaton ϕ and the axion C in the Lunin-Maldacena background [1] for complex β. These read

$$e^{\phi} = e^{\phi_0} G^{1/2} H\,, \qquad C = C^0 + e^{-\phi_0}\hat{\gamma}\hat{\sigma} H^{-1} Q\,, \tag{8.4}$$

$$G^{-1} \equiv 1 + (\hat{\sigma}^2 + \hat{\gamma}^2) Q\ , \qquad H \equiv 1 + \hat{\sigma}^2 Q\,, \tag{8.5}$$

where similarly to the real case, we have defined $\hat{\gamma} := \beta_R g\sqrt{N}$ and $\hat{\sigma} := \sigma g\sqrt{N}$. The function Q is the same as previously, and the axion C is to be distinguished from the constant $C^0 = \frac{\theta}{2\pi}$.

Next we compare the characteristic exponential factor (8.3) arising from the Yang-Mill instanton, to the exponential $e^{2\pi i\tau}$ expected in the modular form terms in the IIB effective action in the Lunin-Maldacena background. We have

$$e^{2\pi i\tau} = e^{2\pi i(ie^{-\phi}+C)} = \exp\left[-2\pi e^{-\phi_0}[1 + \tfrac{1}{2}(\hat{\gamma}^2 - \hat{\sigma}^2)Q] \,+\, 2\pi i(C^0 + e^{-\phi_0}\hat{\gamma}\hat{\sigma} Q)\right] \tag{8.6}$$

In the second equality we have used the expressions for the dilaton and the axion fields of the β-deformed background in the weak-coupling large-N small-β limit, i.e. in the expansion in terms of $\hat{\sigma}$ and $\hat{\gamma}$ we ignore terms of order cubic or higher. By employing the usual relations $\hat{\gamma}^2 = \beta_R^2 N g^2$ and $\hat{\sigma}^2 = \sigma^2 N g^2$ we arrive at

$$e^{2\pi i\tau} \;=\; \exp\left[-\frac{8\pi^2}{g^2} + i\theta \,-\, N(\beta_R^2 - \sigma^2)\, 4\pi^2\, Q \,+\, N 8\pi^2 i\sigma\beta_R Q\right], \tag{8.7}$$

By comparing the last equation to (8.3) it is immediate to see that

$$\mathcal{F} = e^{2\pi i\tau}\,. \tag{8.8}$$

Thus we conclude that our results generalize correctly to the complex β-deformed AdS/CFT correspondence and are in agreement with the expectations on the string theory side.

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A. D-instanton partition function

In this appendix we show construct the D-instanton partition function in the β-deformed string theory and show that it reproduces the corresponding gauge theory result in section 4.

In string theory, D-instanton is a point-like defect — the D(-1) brane, hence its partition function is described in terms of a matrix model integral. The partition function of k D-instantons on N D3-branes in the type IIB theory was previously constructed in ref. [8]. Here we will generalize this construction to include the β-deformation effects on the string theory side.

Following ref. [8] we first consider the kD(-1) branes in interacting with the ND3 branes in the standard type IIB string theory. This description accounts for the D-instanton effects in the undeformed $AdS_5 \times S^5$ background dual to the $\mathcal{N} = 4$ Yang-Mills. From the perspective of the kD(-1) world-volume, the kD(-1)/ND3 brane system is described by the partition function [8],

$$\mathcal{Z}_{k,N} = \int d\mu_{k,N} \, e^{-S_{k,N}} \, . \tag{A.1}$$

Here the D-instanton integration measure $d\mu_{k,N}$ and action $S_{k,N}$ are over the D-instanton collective coordinates, and the D-3 brane degrees of freedom are turned off. This is a $0+0$-dimensional matrix model which can be obtained by the dimensional reduction from the $U(k) \times U(N)$ gauge theory describing kDp branes and ND$(p+4)$ branes.[13] The kDp/ND$(p+4)$ brane system can live in the maximal dimension $p = 5$ which corresponds to the 6-dimensional gauge theory on the world-volume of the D5-branes. Then the cases $5 \geq p \geq -1$ follow by dimensional reduction. The D-instanton partition function corresponds to the minimal case of $p = -1$. For practical calculations it is most convenient to start with the maximal case $p = 5$ to specify the field content of the model, and then reduce to zero dimensions, $p = -1$.

The content of the kD5/ND9 system is described by the $(1,1)$ vector multiplet and two bi-fundamental hypermultiplets in the 6-dimensional world-volume of kD5 branes. The vector multiplet transforms in the adjoint representation of the $U(k)$ gauge group and represents the open-string degrees of freedom of the kD5 branes in isolation. On the other hand, the $U(k) \times U(N)$ bi-fundamental hyper-multiplets incorporate the modes of the open strings stretched between the k branes and the N branes (two species of hypermultiplets correspond to two orientations of the open strings). Thus the hypermultiplets describe the interactions between D-instantons and the spectator branes. The component fields of the vector multiplet are listed in the table 1, and the hypermultiplet fields are listed in table 2.

The D-instanton integration measure is uniquely determined by the action of this $U(k)$ theory with hypermultiplets. Dimensionally reducing from $d = 6$ to 0 dimensions one finds [8] the partition function:

$$\mathcal{Z}_{k,N} = \frac{g_4^4}{\mathrm{Vol}\, U(k)} \int d^{4k^2} a' \, d^{8k^2} \mathcal{M}' d^{6k^2} \chi d^{8k^2} \lambda \, d^{3k^2} D \, d^{2kN} w \, d^{2kN} \bar{w} \, d^{4kN} \mu \, d^{4kN} \bar{\mu} \;\; \exp[-S_{k,N}] \tag{A.2}$$

where $S_{k,N} = g_0^{-2} S_G + S_K + S_D$ and

$$S_G = \mathrm{tr}_k \big(-[\chi_a, \chi_b]^2 + \sqrt{2} i\pi \lambda_{\dot{\alpha}A} [\chi^\dagger_{AB}, \lambda^{\dot{\alpha}}_B] + 2 D^c D^c \big) \, , \tag{A.3a}$$

$$S_K = -\mathrm{tr}_k \big([\chi_a, a'_n]^2 + \chi_a \bar{w}^{\dot{\alpha}}_u w_{u\dot{\alpha}} \chi_a + \sqrt{2} i\pi \mathcal{M}'^{\alpha A} [\chi_{AB}, \mathcal{M}'^B_\alpha] + 2\sqrt{2} i\pi \bar{\mu}^A_u \chi_{AB} \mu^B_u \big) \, , \tag{A.3b}$$

$$S_D = i\pi \mathrm{tr}_k \big([a'_{\alpha\dot{\alpha}}, \mathcal{M}'^{\alpha A}] \lambda^{\dot{\alpha}}_A + \bar{\mu}^A_u w_{u\dot{\alpha}} \lambda^{\dot{\alpha}}_A + \bar{w}_{u\dot{\alpha}} \mu^A_u \lambda^{\dot{\alpha}}_A + \pi^{-1} D^c (\tau^c)^{\dot{\beta}}{}_{\dot{\alpha}} (\bar{w}^{\dot{\alpha}} w_{\dot{\beta}} + \bar{a}'^{\dot{\alpha}\alpha} a'_{\alpha\dot{\beta}}) \big) \, . \tag{A.3c}$$

[13] From the instanton perspective, and in the $\alpha' \to 0$ limit, the overall $U(1)$ factor in the $U(N)$ gauge group is irrelevant. Hence for the purposes of this paper we will not distinguish between the $U(k) \times U(N)$ and $U(k) \times SU(N)$ cases. However, the $U(1)$ factor in the $U(k)$ groups is physically significant, it describes the centre of mass degrees of freedom of the k-instanton which are important.

Component	Description	$U(k)$	$U(N)$
$\chi^{1\ldots 6}$	Gauge Field	$\boldsymbol{k}\times\boldsymbol{k}$	$\mathbf{1}$
$\lambda_{\dot\alpha}$	Gaugino	$\boldsymbol{k}\times\boldsymbol{k}$	$\mathbf{1}$
$D^{1\ldots 3}$	Auxiliary Field	$\boldsymbol{k}\times\boldsymbol{k}$	$\mathbf{1}$
$a'_{\alpha\dot\alpha}$	Scalar Field	$\boldsymbol{k}\times\boldsymbol{k}$	$\mathbf{1}$
$\mathcal{M}'_{\dot\alpha}$	Fermion Field	$\boldsymbol{k}\times\boldsymbol{k}$	$\mathbf{1}$

Table 1: Components of the $(1,1)$ vector multiplet in $d=6$. They describe k D5 branes in isolation.

Component	Description	$U(k)$	$U(N)$
$w_{\dot\alpha}$	Scalar Field	$\boldsymbol{k}$	$\boldsymbol{N}$
μ	Fermion Field	$\boldsymbol{k}$	$\boldsymbol{N}$
$\bar w_{\dot\alpha}$	Scalar Field	$\bar{\boldsymbol{k}}$	$\bar{\boldsymbol{N}}$
$\bar\mu$	Fermion Field	$\bar{\boldsymbol{k}}$	$\bar{\boldsymbol{N}}$

Table 2: Components of bi-fundamental hypermultiplets in $d=6$. They describe interactions between k D5 and N D9 branes.

Equations above define the kD-instanton measure in string theory in the flat background and in presence of the ND-3 branes. We now want to β-deform this background. Lunin and Maldacena have argued in [1] that the open string field theory in the β-deformed background is obtained from the theory on the undeformed background precisely by changing the star-product between the fields carrying the relevant $U(1)$ charges. In our case, this implies that the star product should be used instead of ordinary products for all fields transforming under the $SO(6)=U(4)$ R-symmetry. This requires star products in expressions involving χ, λ and D fields in the equations (A.3a)–(A.3c) above. We also recall that the star product is trivial between the fields of opposite charges and hence can be dropped in the terms which are quadratic in charged fields. This amounts to the following equations for the action terms in (A.2) in the β-deformed background:

$$S_G^\beta = \mathrm{tr}_k\big(-[\chi_a,\chi_b]_*[\chi_a,\chi_b]_* + \sqrt{2}i\pi\lambda_{\dot\alpha A} * [\chi^\dagger_{AB},\lambda^{\dot\alpha}_B]_* + 2D^cD^c\big)\,, \tag{A.4a}$$

$$S_K^\beta = -\mathrm{tr}_k\big([\chi_a,a'_n]^2 + \chi_a\bar w^{\dot\alpha}_u w_{u\dot\alpha}\chi_a + \sqrt{2}i\pi\mathcal{M}'^{\alpha A} * [\chi_{AB},\mathcal{M}'^B_\alpha]_* + 2\sqrt{2}i\pi\bar\mu^A_u * \chi_{AB} * \mu^B_u\big)\,, \tag{A.4b}$$

$$S_D^\beta = i\pi\mathrm{tr}_k\big([a'_{\alpha\dot\alpha},\mathcal{M}'^{\alpha A}]\lambda^{\dot\alpha}_A + \bar\mu^A_u w_{u\dot\alpha}\lambda^{\dot\alpha}_A + \bar w_{u\dot\alpha}\mu^A_u\lambda^{\dot\alpha}_A + \pi^{-1}D^c(\tau^c)^{\dot\beta}{}_{\dot\alpha}(\bar w^{\dot\alpha}w_{\dot\beta} + \bar a'^{\dot\alpha\alpha}a'_{\alpha\dot\beta})\big) = S_D \tag{A.4c}$$

The D-instanton partition function $\mathcal{Z}_{k,N}$ depends explicitly on the inverse string tension α' through the zero-dimensional coupling $g_0^2 \propto (\alpha')^{-2}$ which appears in $g_0^{-2} S_G^\beta$ which comes from the dimensional reduction of the $d = 6$ gauge action. In the field theory limit the fundamental string scale is set to zero, $\alpha' = 0$, to decouple the world-volume gauge theory from gravity. Thus, as explained in [8], to derive the ADHM-instanton measure in conventional supersymmetric gauge theory one must take the limit $\alpha' \to 0$. In this limit $g_0^2 \to \infty$ equations of motion for D^c are precisely the non-linear ADHM constraints, first equation in (4.5). Similarly equations of motion for λ are the fermionic ADHM constraints in (4.5). Integration over D^c and $\lambda^{\dot\alpha}_A$ yields δ-functions which impose the constraints.

We can now make contact with our result (4.15) for the instanton partition function in gauge theory derived in section 4. First, we integrate out the D^c and $\lambda^{\dot\alpha}_A$ variables, thus getting the δ-functions of the ADHM constraints, precisely as in (4.15). Second, we rewrite S_K^β as,

$$S_K^\beta = \operatorname{tr}_k \chi_a \mathbf{L} \chi_a - 4\pi i \operatorname{tr}_k \chi_{AB} \Lambda_{A*B} . \tag{A.5}$$

This is equal to (minus) the exponent appearing in equation (4.12). On integrating out the gauge field χ_a, the instanton action reduces to the fermion quadrilinear term (4.9). We have therefore reproduced our result for the ADHM measure in the β-deformed gauge theory, up to an overall normalization constant. This is completely analogous to the matching between D- and gauge-instanton partition functions discovered in [8] – the only novelty in the present case is the appearance of the star products on both sides of the correspondence. What this matching really tests in the β-deformed theory is the validity of the prescription for introducing β-deformations in the open-string theory conjectured by Lunin and Maldacena and which we have used to derive the results (A.4a)–(A.4c) above.

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